

A thermodynamic geometric study of Rényi and Tsallis entropies

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A general investigation is made into the intrinsic Riemannian geometry for complex systems, from the perspective of statistical mechanics. The entropic formulation of statistical mechanics is the ingredient which enables a connection between statistical mechanics and the corresponding Riemannian geometry. The form of the entropy used commonly is the Shannon entropy. However, for modelling complex systems, it is often useful to make use of entropies such as the Rényi and Tsallis entropies. We consider, here, Shannon, Rényi, Tsallis, Abe and structural entropies, for our analysis. We focus on one, two and three particle thermally excited configurations. We find that statistical pair correlation functions, determined by the components of the covariant metric tensor of the underlying thermodynamic geometry, associated with the various entropies have well defined, definite expressions, which may be extended for arbitrary finite particle systems. In all cases, we find a non-degenerate intrinsic Riemannian manifold. In particular, any finite particle system described in terms of Rényi, Tsallis, Abe and structural entropies, always corresponds to an interacting statistical system, thereby highlighting their importance in the study of complex systems. On the other hand, a statistical description by the Gibbs-Shannon entropy corresponds to a non-interacting system.

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I. INTRODUCTION

Entropy is one of the cornerstones of statistical mechanics, and is used extensively in many studies. Apart from the usual Gibbs-Shannon entropy, there is a growing consensus that for an understanding of complex systems, it would be useful to go beyond the Shannon entropy. For this reason, the Tsallis [1] and Rényi entropies [2, 3], have been used extensively. On the other hand, an elegant geometrical formulation of thermodynamics has been developed, see [4] for a review. In this, the theory of thermodynamic fluctuations is developed from a macroscopic perspective, making use of the notions of covariance and consistency, expressed naturally using the language of Riemannian geometry.

Motivated by a need to understand complex systems, we make a study of the intrinsic Riemannian geometry associated with the usual Gibbs-Shannon as well as the Rényi and Tsallis entropies. In addition, we also study the Abe [5] and the so called structural [6] entropies. This provides a geometric insight into the systems, studied using these entropies. Thus, for example, the components of the covariant metric tensor, of the underlying Riemannian geometry of the thermodynamic phase space, provides insight into the local correlations inherent in the system, which could be used to study its local stability. On the other hand, the corresponding thermodynamic scalar curvature is a signature of the global correlations present, and could be used to address issues related to phase transitions, in the system.

The plan of the paper is as follows. In Section II, we make a brief review of thermodynamic geometry. Thermodynamic geometry is applied in Section III, to a study of various entropies, for few particle systems. The entropies studied are the Shannon entropy, as well as the Rényi, Tsallis, Abe and structural entropies. This is followed up in Section IV, by a geometrical interpretation of the additivity of Rényi and pseudo-additivity of Tsallis entropies. Finally, in Section V, we make our Conclusions.

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II. THERMODYNAMIC GEOMETRY: A BRIEF REVIEW

This section provides a brief account of Ruppeiner geometry, which as an intrinsic Riemannian manifold (M, g) , serves the purpose of describing the nature of statistical fluctuations in an open thermodynamic configuration. Before focussing our attention on a specific configuration, it is worthwhile reviewing some of the basics of intrinsic Riemannian geometry. More details can be found in [4, 7–14] for black holes in general relativity, [15–18] for a recent account of these notions in string theory and M-theory and [19] for associated chemical correlations and quark number susceptibilities in 2- and 3-flavor hot QCD configurations.

In what follows, we shall consider an intrinsic Riemannian geometric model whose covariant metric tensor may be defined as the Hessian matrix of the entropy with respect to an arbitrary number of parameters characterizing the thermodynamics of the system of interest. We shall focus our attention on the parameters, such as fixed volume, characterizing the thermodynamics of the equilibrium configuration. In particular, let us define a representation $S(q, p_i)$ for a given entropy, generalized temperature, and probabilities $\{S, q, p_i\}$. Here p_i is the probability that a given event i has occurred, and would be determined, for an equilibrium distribution, by its temperature. These p_i 's, for $i = 1, \dots, n$, when treated as a set of extensive thermodynamic variables, form coordinate charts for the corresponding intrinsic manifold. Also, an appropriate choice of the parameter q , characterizes the entropy of the system.

The probability distribution of thermodynamic fluctuations, in an equilibrium intrinsic space, naturally characterizes the invariant interval of the corresponding thermodynamic geometry. It can be shown that the probability distribution in the Gaussian approximation is

$$\Omega(x) = A \exp \left(-\frac{1}{2} g_{ij}(x) dx^i \otimes dx^j \right), \quad (1)$$

where the pre-factor A is the normalization constant and \otimes signifies symmetric product of the Gaussian distribution. The associated inverse metric may thus be shown to be the second moment of the quadratic fluctuations or the pair correlation functions. An explicit evaluation shows the components of the inverse metric tensor to be

$$g^{ij} = \langle X^i | X^j \rangle, \quad (2)$$

where $\{X_i\}$'s are the extensive thermodynamic variables conjugate to the intensive variables $\{x_i\}$. Moreover, such Riemannian structures may as well be expressed in terms of any suitable thermodynamic potential obtained by Legendre transform(s), which in our intrinsic geometric set-up corresponds to certain general coordinate transformations on the equilibrium statistical configuration.

We shall apply the above formalism, in this work, to a fixed number of particles. This method, in general, implies that the quadratic fluctuations in an underlying statistical system may be described by the formalism of intrinsic Riemannian geometry, which is solely based on nothing other than the relative probabilities and generalized temperature. This, in turn, implies that the space spanned by the n parameters of the system under consideration exhibits a n -dimensional intrinsic Riemannian manifold (M_n, g) , the so called thermodynamic geometry [4]. The components of the covariant metric tensor may be defined as

$$g_{ij} := -\frac{\partial^2 S(\vec{X})}{\partial X^i \partial X^j}, \quad (3)$$

where the vector $\vec{X} = (q, p^i) \in M_n$. Explicitly, for the case of two dimensional intrinsic geometry parametrized by $\vec{X} = (p^1, p^2) \in M_2$, the components of the thermodynamic metric tensor are given by

$$g_{p_i p_j} = -\frac{\partial^2 S}{\partial p_i \partial p_j} \equiv S_{p_i p_j}, \quad (4)$$

where, $i, j = 1, 2$. In this case the determinant of the metric tensor may be expressed as,

$$\|g\| = S_{p_1 p_1} S_{p_2 p_2} - S_{p_1 p_2}^2. \quad (5)$$

The Christoffel connection Γ_{ijk} , Riemann curvature tensor R_{ijkl} , Ricci tensor R_{ij} and scalar curvature R for the two dimensional thermodynamic geometry (M_2, g) can be computed. The scalar curvature can be shown to be

$$R = \frac{1}{2} \left(S_{p_1 p_1} S_{p_2 p_2} - S_{p_1 p_2}^2 \right)^{-2} \left(S_{p_2 p_2} S_{p_1 p_1 p_1} S_{p_1 p_2 p_2} \right)$$

$$\begin{aligned}
& +S_{p_1 p_2} S_{p_1 p_1 p_2} S_{p_1 p_2 p_2} + S_{p_1 p_1} S_{p_1 p_1 p_2} S_{p_2 p_2 p_2} \\
& -S_{p_1 p_2} S_{p_1 p_1 p_1} S_{p_2 p_2 p_2} - S_{p_1 p_1} S_{p_1 p_2 p_2}^2 \\
& -S_{p_2 p_2} S_{p_1 p_1 p_2}^2 \Big). \tag{6}
\end{aligned}$$

Interestingly, the relation between the thermodynamic scalar curvature and the Riemann curvature tensor for the two dimensional intrinsic geometry is given (see for details [14, 15]) by

$$R = \frac{2}{\|g\|} R_{p_1 p_2 p_1 p_2}. \tag{7}$$

One may further see that this relation is quite natural for any intrinsic Riemannian surface $(M_2(R), g)$. Multi-parameter statistical configurations can also be studied along similar lines. Physically, it is intriguing to note that the Riemannian structure defined by the metric tensor, in the entropy representation, is rather closely related to classical thermodynamic fluctuation theory [20] and critical phenomena, if any. Even though the above geometric formulation thus tacitly involves a microscopic basis in terms of a chosen ensemble, nevertheless, the present analysis has only been considered in the thermodynamic limit of open statistical systems.

It is worth noting that the relation of a non-zero scalar curvature with an underlying interacting statistical system remains valid for higher dimensional intrinsic manifolds and the connection of a divergent (scalar) curvature with phase transitions may as well be analyzed from the Hessian matrix of the considered fluctuating entropy. Our analysis thus takes into account scales that are larger than the correlation length where a few microstates cannot dominate the entire macroscopic phase-space configuration.

We shall focus on the interpretation that the underlying entropy includes contributions from a large number of microstates and make use of our description of the geometric thermodynamics of open systems. With this general introduction to the thermodynamic geometry defined as the Hessian function of the configurational entropy, we shall now proceed to investigate the thermo-geometric structures of finite particle systems, with few parameters. We shall compute entropies and study the corresponding thermodynamic geometry.

Having provided a brief description of thermodynamic geometry, which gives clear prescription for obtaining the metric tensor $\|g\|$ and associated Ricci scalar curvature R , it is worth mentioning that these quantities find relevance in diverse studies, since the scalar curvature represents the correlation length of the chosen statistical configuration [4, 9]. This would be important, as we shall show below, in the context of complex statistical systems which are seen to be generically interacting.

III. THERMODYNAMIC GEOMETRICAL APPROACH TO COMPLEX SYSTEMS

What is presented here may be motivated by saying that a system can be described by, say, the two states of a coin, *viz.*, heads H or tail T . This would be called, in our nomenclature, a two particle system, the classical analog of a two-level quantum system (qubit), having two probabilities, taking values over $\{H, T\}$. Similarly, a three particle system can be described by three probabilities, and would be the classical analog of a three-level system. For an entropic description of complex systems, we shall use the Shannon entropy, as a comparison with, among others, Tsallis and Rényi entropies [1–3]. We shall consider equilibrium thermodynamic systems, and describe their nature under Gaussian fluctuations. It will be shown that the symmetric nature of fluctuations define an intrinsic Riemannian geometry, whose metric tensor may be obtained as the negative Hessian of the associated entropy. The number of concerned parameters of a chosen configuration basically determines the dimensionality of phase-space.

A. Shannon Entropy

Here we present a brief account of the essential features of thermodynamic geometries from the perspective of multi-parameter Shannon configurations. In particular, we shall focus our attention on the geometric nature of the underlying ensemble having finitely many particles. The expression for Shannon entropy, for arbitrary number of probabilities, is given by

$$S = - \sum_i p_i \ln p_i. \tag{8}$$

Below, we shall compute the concerned configurations for an arbitrary $i = 2, 3, \dots$ and then analyze the generic nature of statistical correlations in them. The number of probabilities define the dimension of the underlying thermodynamic configuration.

1. Two particle Shannon system

Let us first consider the simplest case of Shannon configuration with two independent probabilities $\{p_1, p_2\}$. In this case, the entropy may be expressed as

$$S(p_1, p_2) := -p_1 \ln(p_1) - p_2 \ln(p_2). \quad (9)$$

The components of the covariant metric tensor are explicitly given by

$$g_{ij} = \frac{1}{p_i} \delta_{ij}, \quad i, j = 1, 2 \quad (10)$$

For the two particle Shannon configuration, we find that the determinant of the metric tensor is non-zero and varies as inverse of the either of the probabilities. In particular, we find that the determinant of the metric tensor is

$$\|g\| = \frac{1}{p_1 p_2}. \quad (11)$$

Furthermore, it is not difficult to see that the scalar curvature vanishes, $R = 0$, and thus the underlying statistical basis is a non-interacting configuration.

2. Three particle Shannon system

An addition of third probability as an intrinsic coordinate to the foregoing configuration shows that the expression of entropy in this case is

$$S(p_1, p_2, p_3) := -p_1 \ln(p_1) - p_2 \ln(p_2) - p_3 \ln(p_3). \quad (12)$$

We may easily observe that the components of the covariant metric tensor are

$$g_{ij} = \frac{1}{p_i} \delta_{ij}; \quad i, j = 1 - 3. \quad (13)$$

A similar analysis finds non-zero determinant of the metric tensor, which takes the form

$$\|g\| = \frac{1}{p_1 p_2 p_3}. \quad (14)$$

In this case, again, we find that the scalar curvature identically vanishes, with $R = 0$. Thus, we find that the underlying three particle Shannon configurations are well-defined and imply non-interacting statistical system.

3. Multi-particle Shannon system

We further notice the same conclusions for any finitely many particle Shannon statistical system. In general, we see for multi-particle Shannon systems that the Shannon pair correlation functions are

$$g_{ij} = \frac{1}{p_i} \delta_{ij}; \quad i, j = 1 - n. \quad (15)$$

It is worth mentioning that the determinant of the metric tensor is

$$\|g\| = \left(\prod p_i \right)^{-1}. \quad (16)$$

An easy inspection finds, for all $i = 2, 3, \dots$, that the corresponding statistical configurations have zero scalar curvature and are non-interacting. Our analysis thus shows that the general Shannon configurations are well-defined thermodynamic systems and have an identically vanishing correlation lengths.

B. Rényi Entropy

In this subsection, we shall use the essential features of thermodynamic geometry to describe systems using Rényi entropy, with an increasing number of parameters. Here, we focus our attention on the geometric nature of the local and global correlations in the neighbourhood of small fluctuations in the chosen Rényi configurations. As stated earlier, the thermodynamic metric in the probability space is given by the negative Hessian matrix of the entropy with respect to the variables defining the space, which here, would be the two, three,... distinct probabilities carried by the Rényi configurations. The entropy of n -particle Rényi system [3] is given by

$$S_q^R = \frac{1}{1-q} \ln \left(\sum_i p_i^q \right). \quad (17)$$

In order to analyze the underlying interactions, we shall proceed by considering, first, the nature of single particle system at given finite temperature.

1. Single particle Rényi system

We shall first consider a single particle Rényi system at temperature T . From the Eq. (17), we then see that the corresponding entropy reduces to

$$S_q^R(q, p_1) := \frac{1}{1-q} \ln(p_1^q), \quad (18)$$

Following Eq. (4), a straightforward calculation Eq. shows that the components of the covariant metric tensor are

$$\begin{aligned} g_{qq} &= 2 \left(\frac{\ln(p_1)q - \ln(p_1^q) - \ln(p_1)}{(1-q)^3} \right), \\ g_{qp_1} &= -\frac{1}{(1-q)^2 p_1}, \\ g_{p_1 p_1} &= \frac{1}{(1-q)p_1^2}. \end{aligned} \quad (19)$$

Applying Eq. (5), the determinant of the metric tensor can be shown to have the simple form

$$\|g\| = \frac{2 \ln(p_1)q^2 - 2q \ln(p_1^q) - 2 \ln(p_1)q - 1}{(1-q)^4 p_1^2}. \quad (20)$$

Finally, it is not difficult to see from Eq. (6), that the Ricci scalar is

$$R = \frac{(-1+q)(\ln(p_1) + \ln(p_1^q) - 2 \ln(p_1)q - q \ln(p_1^q) + \ln(p_1)q^2 + 1 - 2q)}{(2q \ln(p_1^q) + 2 \ln(p_1)q - 2 \ln(p_1)q^2 + 1)^2}. \quad (21)$$

Note that the extreme limit of the system is described with the Rényi parameter $q := 1$; and then one finds a non-interacting configuration with $R = 0$, while the determinant of the metric tensor $\|g\|$ diverges. The free particle is described by the Rényi parameter $q := \frac{1}{3}$ (this value of q corresponds to a one dimensional ideal gas [3]). Here, it is easy to show that the geometric quantities reduce to their corresponding limiting values. In particular, the scalar curvature is seen to be

$$R = -\frac{2}{3} \left(\frac{\frac{4}{9} \ln(p_1) + \frac{2}{3} \ln(p_1^{1/3}) + \frac{1}{3}}{(\frac{4}{9} \ln(p_1) + \frac{2}{3} \ln(p_1^{1/3}) + 1)^2} \right), \quad (22)$$

and the determinant of the metric tensor is

$$\|g\| = \frac{81}{16p_1^2} \left(-\frac{4}{9} \ln(p_1) - \frac{2}{3} \ln(p_1^{1/3}) - 1 \right). \quad (23)$$

The determinant of the metric (DM) and the scalar curvature for the single particle Rényi configuration are shown in Figs. (1) and (2), respectively. From Fig. (1), it is clear that DM is non-zero and non-negative for the chosen range of parameters and diverges only near $p = 0, q = 0$. The curvature scalar does not show any divergence in general except at the zeros of the determinant (see Fig. (2)). From fig. (2), the nature of the thermodynamic curvature of the Rényi entropy, that is the Rényi correlation length, can be read-off for the case of free particle $q := 1/3$, as well as for more general cases. The bumps in curvature show the presence of non-trivial interactions in the statistical configuration. Larger the height of a bump, stronger will be the interaction.

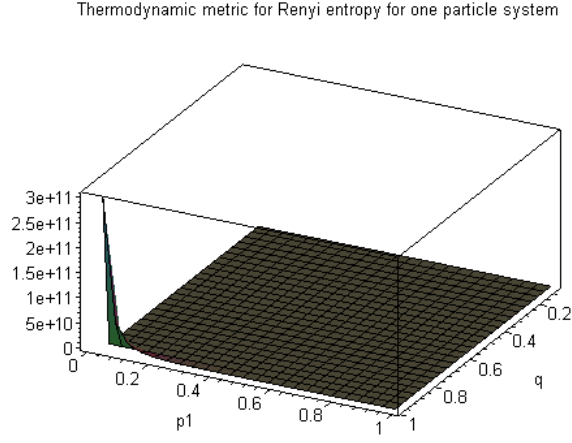


FIG. 1: The determinant of the metric tensor as a function of probability p , and the entropic parameter q (17), in a single particle Rényi system.

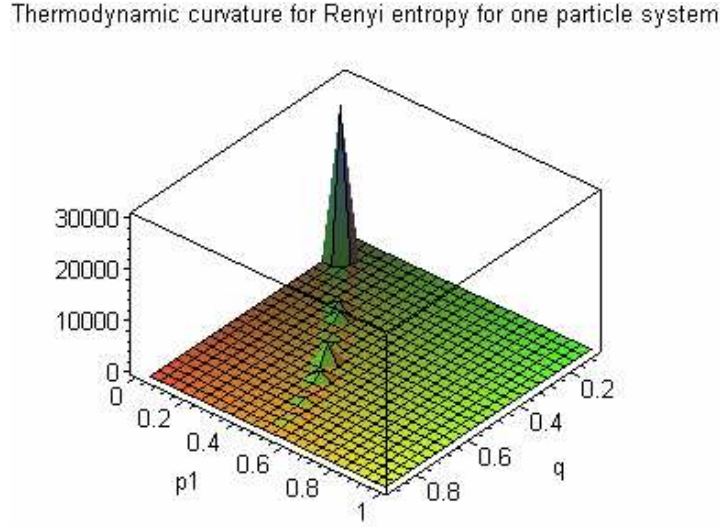


FIG. 2: The scalar curvature as a function of probability p , and the entropic parameter q (17), in a single particle Rényi system.

2. Two particle Rényi system

From Eq. (17), we see that the expression for the entropy of the two particle Rényi configuration reduces to

$$S_q^R(q, p_1, p_2) := \frac{1}{1-q} \ln(p_1^q + p_2^q). \quad (24)$$

The components of corresponding metric tensor are

$$\begin{aligned} g_{qq} = & \left(-2p_1^{2q} \ln(p_1)q + 4 \ln(p_1^q + p_2^q) p_1^q p_2^q + p_2^q \ln(p_2)^2 p_1^q \right. \\ & + p_1^q \ln(p_1)^2 p_2^q + 2p_2^q \ln(p_2) p_1^q + 2p_1^q \ln(p_1) p_2^q - 2p_2^{2q} \ln(p_2)q \\ & - 2p_1^q \ln(p_1) p_2^q \ln(p_2) + p_2^q \ln(p_2)^2 q^2 p_1^q - 2p_2^q \ln(p_2)^2 q p_1^q \\ & + p_1^q \ln(p_1)^2 q^2 p_2^q - 2p_1^q \ln(p_1)^2 q p_2^q - 2p_2^q \ln(p_2) q p_1^q \\ & \left. - 2p_1^q \ln(p_1) q p_2^q + 2p_2^{2q} \ln(p_2) + 2p_1^{2q} \ln(p_1) + 2 \ln(p_1^q + p_2^q) p_1^{2q} \right) \end{aligned}$$

$$\begin{aligned}
& +2 \ln(p_1^q + p_2^q) p_2^{2q} - 2p_1^q \ln(p_1) p_2^q \ln(p_2) q^2 \\
& + 4p_1^q \ln(p_1) p_2^q \ln(p_2) q \Big) \times ((p_1^q + p_2^q)^2 (-1 + q)^3)^{-1}.
\end{aligned} \tag{25}$$

Explicitly, we notice that the components of the metric tensor satisfy

$$g_{qq}(q, p_1, p_2) = g_{qq}(q, p_2, p_1). \tag{26}$$

Furthermore, there exists a symmetric expression for the components of thermodynamic metric with one p_i , and in particular it is seen that

$$\begin{aligned}
g_{qp_i} = & - \left(p_i^q \ln(p_i) q p_j^q - \ln(p_i) q^2 p_j^q p_i^q + p_i^{2q} + p_i^q p_j^q - p_j^q \ln(p_j) q p_i^q \right. \\
& \left. + p_i^q p_j^q \ln(p_j) q^2 \right) \times ((-1 + q)^2 p_i (p_i^q + p_j^q)^2)^{-1},
\end{aligned} \tag{27}$$

for $i, j = 1, 2$. Similarly, the diagonal components of the metric tensor are

$$g_{p_i p_i} = q \left(p_i^{q-2} q p_j^q - p_i^{2q-2} - p_i^{q-2} p_j^q \right) \times ((-1 + q)(p_i^q + p_j^q)^2)^{-1}, \tag{28}$$

for $i, j = 1, 2$. Finally, the off-diagonal components turn out to be

$$g_{p_i p_j} = -(p_i^{q-1} q^2 p_j^{q-1}) ((-1 + q)(p_i^q + p_j^q)^2)^{-1}, \tag{29}$$

for $i \neq j$. A straightforward computation shows that the determinant of the metric tensor is

$$\begin{aligned}
\|g\| = & -q \left(2p_1^q p_2^{3q} \ln(p_2) q - 2p_1^q p_2^{3q} \ln(p_2) q^2 + p_1^q p_2^{3q} + 2p_1^q p_2^{3q} \ln(p_1^q + p_2^q) q + 4p_2^{2q} \ln(p_1^q + p_2^q) p_1^{2q} q \right. \\
& + p_2^{2q} \ln(p_2)^2 q p_1^{2q} - 2 \ln(p_1) q^2 p_2^{2q} p_1^{2q} + 2p_1^{2q} \ln(p_1) q p_2^{2q} - 2p_1^{2q} \ln(p_1) p_2^{2q} \ln(p_2) q + p_1^{2q} \ln(p_1)^2 q p_2^{2q} \\
& + 2p_1^{2q} \ln(p_1) p_2^{2q} \ln(p_2) q^2 - 2p_1^{2q} p_2^{2q} \ln(p_2) q^2 - p_1^{2q} \ln(p_1)^2 q^2 p_2^{2q} + 2p_2^{2q} \ln(p_2) q p_1^{2q} - p_2^{2q} \ln(p_2)^2 q^2 p_1^{2q} \\
& \left. + 2p_1^{2q} p_2^{2q} + 2p_2^q p_1^{3q} \ln(p_1) q - 2p_2^q \ln(p_1) q^2 p_1^{3q} + p_2^q p_1^{3q} + 2p_2^q \ln(p_1^q + p_2^q) p_1^{3q} q \right) \times \\
& (p_1^2 p_2^2 (p_1^q + p_2^q)^4 (-1 + q)^4)^{-1}.
\end{aligned} \tag{30}$$

The scalar curvature does not have a very illuminating form, in general. Nevertheless, for the case of equal probabilities, that is with $p_1 = p_2 = p$, the curvature is illustrated in Fig. (4). In this case, the statistical system corresponds to an ensemble of free particles, described by the Rényi parameter $q := 1/3$ and the scalar curvature is

$$R = \frac{3}{2} \left(\frac{\frac{640}{81} \ln(2p^{1/3}) - \frac{1280}{243} \ln(p) - \frac{64}{9}}{(\frac{8}{3} \ln(2p^{1/3}) + \frac{16}{9} \ln(p) + 4)^2} \right). \tag{31}$$

From Fig. (3), it is clear that DM is non-zero and non-negative for the chosen range of parameters and diverges only near $p = 0, q = 0$. The plots of Rényi correlation length are shown in Fig. (4) and can be easily analyzed for (a) free particle case with $q := 1/3$, showing no bumps, and (b) in general, showing bumps in some places, corresponding to thermodynamic interactions in the underlying system.

3. Three particle Rényi system

In order to further understand the nature of generic Rényi configurations, we shall now consider a three particle system. The associated entropy, following Eq. (17), is

$$S_q^R(q, p_1, p_2) := \frac{1}{1-q} \ln(p_1^q + p_2^q + p_3^q). \tag{32}$$

Thermodynamic metric for Renyi two particle system

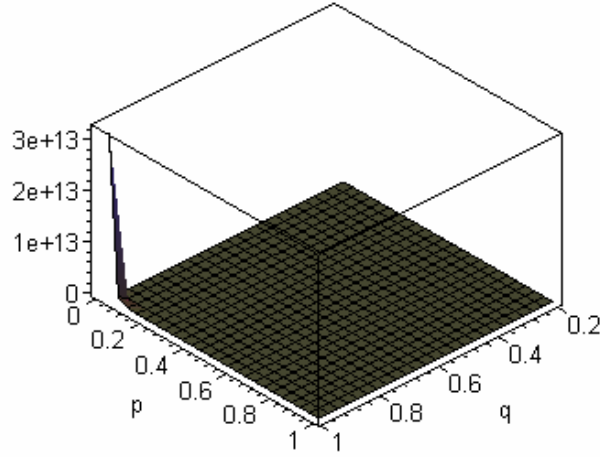


FIG. 3: The determinant of the metric tensor as a function of probability p (both the particle posses equal probability), and the entropic parameter q , in a two particle Rényi system

Thermodynamic curvature for Renyi two particle system

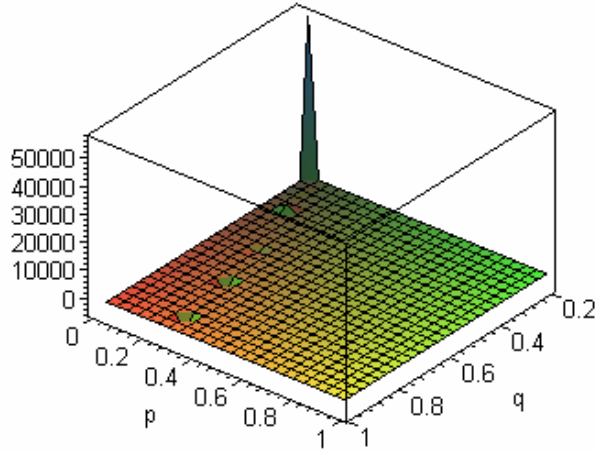


FIG. 4: Thermodynamic curvature as a function of probability p , and the entropic parameter q , in a two particle Rényi system

All observations made in the previous sub-section may be generalized for arbitrary finitely many particle Rényi configuration, and in particular, the components of the metric tensor for the three particle system can be seen to be

$$\begin{aligned}
 g_{qq} = & \left(-2p_3^q \ln(p_3)^2 qp_1^q + p_3^q \ln(p_3)^2 q^2 p_1^q - 2\ln(p_1) qp_1^q p_2^q \right. \\
 & + 4p_1^q \ln(p_1) p_2^q \ln(p_2) q - 2p_2^q \ln(p_2) p_3^q \ln(p_3) q^2 + 2p_3^q \ln(p_3) p_2^q \\
 & + 2p_2^q \ln(p_2) p_3^q + 2p_1^q \ln(p_1) p_2^q + 2p_1^q \ln(p_1) p_3^q + 2p_2^q \ln(p_2) p_1^q \\
 & - 2p_1^q \ln(p_1) p_3^q \ln(p_3) q^2 + 2p_3^q \ln(p_3) p_1^q - 2p_1^q \ln(p_1) p_2^q \ln(p_2) q^2 \\
 & - 2p_3^q \ln(p_3) qp_1^q - 2p_2^q \ln(p_2) qp_1^q - 2p_1^q \ln(p_1) qp_1^q p_3^q \\
 & - 2p_2^q \ln(p_2) qp_1^q + p_2^q \ln(p_2)^2 q^2 p_1^q - 2\ln(p_1)^2 qp_1^q p_2^q \\
 & + \ln(p_1)^2 q^2 p_1^q p_3^q - 2\ln(p_1)^2 qp_1^q p_3^q + \ln(p_1)^2 q^2 p_1^q p_2^q + 2p_1^{2q} \ln(p_1) \\
 & \left. - 2\ln(p_1) qp_1^{2q} + p_3^q \ln(p_3)^2 p_2^q + p_1^q \ln(p_1)^2 p_2^q + p_2^q \ln(p_2)^2 p_1^q \right)
 \end{aligned}$$

$$\begin{aligned}
& +p_1^q \ln(p_1)^2 p_3^q + p_3^q \ln(p_3)^2 p_1^q + p_2^q \ln(p_2)^2 p_3^q \\
& +4 \ln(p_1^q + p_2^q + p_3^q) p_1^q p_2^q - 2p_1^q \ln(p_1) p_2^q \ln(p_2) \\
& -2p_1^q \ln(p_1) p_3^q \ln(p_3) - 2p_2^q \ln(p_2) p_3^q \ln(p_3) \\
& +4 \ln(p_1^q + p_2^q + p_3^q) p_1^q p_3^q - 2p_2^q \ln(p_2) q p_3^q - 2p_3^q \ln(p_3) q p_2^q \\
& +p_3^q \ln(p_3)^2 q^2 p_2^q - 2p_3^q \ln(p_3)^2 q p_2^q + p_2^q \ln(p_2)^2 q^2 p_3^q \\
& -2p_2^q \ln(p_2)^2 q p_3^q + 4p_1^q \ln(p_1) p_3^q \ln(p_3) q + 4p_2^q \ln(p_2) p_3^q \ln(p_3) q \\
& +2p_3^{2q} \ln(p_3) + 2p_2^{2q} \ln(p_2) - 2p_3^{2q} \ln(p_3) q - 2p_2^{2q} \ln(p_2) q \\
& +4 \ln(p_1^q + p_2^q + p_3^q) p_2^q p_3^q + 2 \ln(p_1^q + p_2^q + p_3^q) p_2^{2q} \\
& +2 \ln(p_1^q + p_2^q + p_3^q) p_1^{2q} + 2 \ln(p_1^q + p_2^q + p_3^q) p_3^{2q} \Big) \times \\
& ((-1+q)^3 (p_1^q + p_2^q + p_3^q)^2)^{-1}.
\end{aligned} \tag{33}$$

We observe that for all i and j , the metric tensors are pair-wise symmetric, that is under the exchange of $\{i, j\}$ we have

$$g_{qq}(q, p_i, p_j) = g_{qq}(q, p_j, p_i). \tag{34}$$

Also

$$\begin{aligned}
g_{qp_i} = & \left(-\ln(p_i) q p_i^q p_j^q - \ln(p_i) q p_i^q p_k^q + \ln(p_i) q^2 p_i^q p_j^q + \ln(p_i) q^2 p_i^q p_k^q \right. \\
& -p_i^{2q} - p_i^q p_j^q - p_i^q p_k^q + p_j^q \ln(p_j) q p_i^q - p_j^q \ln(p_j) q^2 p_i^q \\
& \left. + p_k^q \ln(p_k) q p_i^q - p_k^q \ln(p_k) q^2 p_i^q \right) \times ((-1+q)^2 p_i (p_i^q + p_j^q + p_k^q)^2)^{-1},
\end{aligned} \tag{35}$$

for $i, j = 1, 2, 3$ and $j \neq k$. The function g_{qp_i} is symmetric under the exchange between p_j and p_k . Furthermore,

$$g_{p_i p_i} = q \left(p_i^{q-2} q p_j^q + p_i^{q-2} q p_k^q - p_i^{2q-2} - p_i^{q-2} p_j^q - p_i^{q-2} p_k^q \right) \times ((-1+q)(p_i^q + p_j^q + p_k^q)^2)^{-1}, \tag{36}$$

for $i, j = 1, 2, 3$ and $j \neq k$. The functions $g_{p_i p_i}$ are also symmetric in p_j and p_k . As before, it can be seen that for $i \neq j$, the distinct components of the metric tensor are

$$g_{p_i p_j} = -(p_i^{q-1} q^2 p_j^{q-1}) ((-1+q)(p_i^q + p_j^q + p_k^q)^2)^{-1}. \tag{37}$$

A comparison between Eqs. (37) and (29) shows that the form of the metric tensor, for the above distinct three particle case, is identical with that of a two particle Rényi configuration. In this case, a computation for the equiprobable configurations, with $p_1 = p_2 = p_3 = p$, shows that the determinant of the metric tensor reduces to

$$\|g\| = -\frac{1}{27p^6} \left(\frac{q^2 (2 \ln(p) q - 2q^2 \ln(p) + 1 + 2 \ln(3p^q) q)}{(1-q)^4} \right). \tag{38}$$

For non interacting Rényi parameter $q := 1/3$; we may easily see that the scalar curvature take the value of

$$\begin{aligned}
R = & \frac{3}{2} \left(45p^{4/3} + 20p^{4/3} \ln(3p^{1/3}) + \frac{40}{3} \ln(p) p^{4/3} + \frac{16}{9} \ln(p)^2 p^{4/3} + \frac{16}{3} \ln(p) p^{4/3} \ln(3p^{1/3}) \right. \\
& \left. + 4p^{4/3} \ln(3p^{1/3})^2 \right) \times \left(4 \ln(p) p^{2/3} + 9p^{2/3} + 6p^{2/3} \ln(3p^{1/3}) \right)^{-2}.
\end{aligned} \tag{39}$$

Finally, the extreme value $q := 1$, leads to a degenerate metric whose determinant has a division by zero. However, interestingly, we find in this case a non-negative scalar curvature,

$$R = \frac{1}{2} \left(\frac{81p^4 + 324p^4 \ln(3p) + 324p^4 \ln(3p)^2}{(9p^2 + 18p^2 \ln(3p))^2} \right). \tag{40}$$

Here, we find the surprising fact that the three particle Rényi configuration has a different geometric nature, and the degenerate metric has an interaction. In contrast to the one and two particle systems, the three particle system shows larger attraction (negative curvature) (Fig. (6)). An explanation of the origin of these interactions may lie in some form of superstatistics [21, 22], where the entropic parameter q is defined by physical properties of a complex system, that can exchange energy and heat with a thermostat. From Fig. (5), as in the previous cases, the DM is non-zero and non-negative for the chosen range of parameters and diverges near $(p, q) = (0, 0)$.

Thermodynamic metric for Renyi three particle system

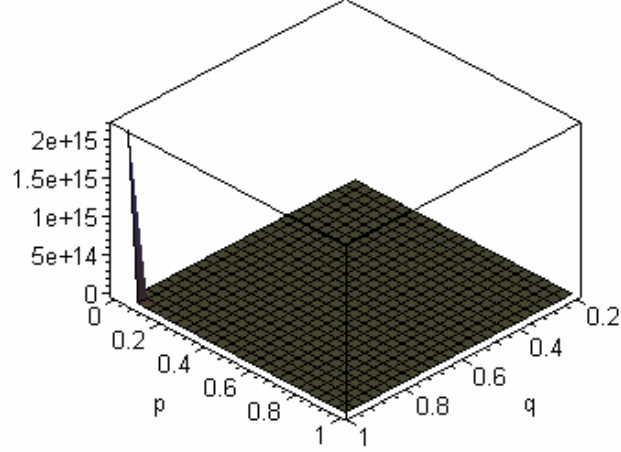


FIG. 5: The determinant of the metric tensor as a function of probability p , and the entropic parameter q , in a three particle Rényi system

Thermodynamic curvature for Renyi three particle system

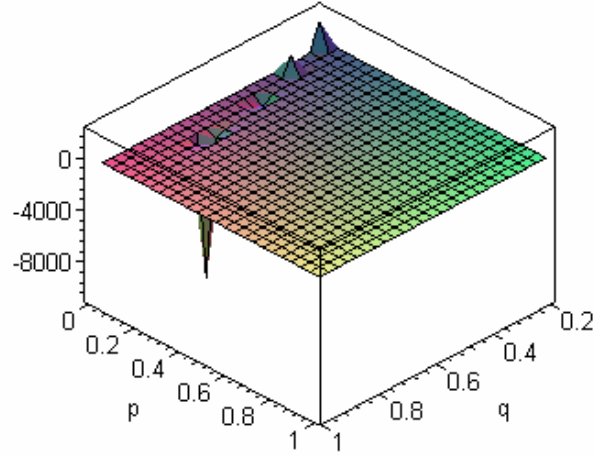


FIG. 6: Thermodynamic curvature as a function of probability p , and the entropic parameter q , in a three particle Rényi system

C. Tsallis Entropy

We now turn our attention to the Tsallis entropy [1]. Its general form is given by [2]

$$S_q^T = -\frac{1}{1-q} \left(1 - \sum_i p_i^q \right). \quad (41)$$

As in the previous subsection, we shall focus our attention on increasing number of particles. We begin with a single particle at temperature T .

1. Single particle Tsallis system

For a single particle Tsallis configuration at the given temperature, the entropy as a function of relative probability p_1 and entropic parameter q is given by

$$S_q^T(q, p_1) = -\frac{1}{1-q}(1 - p_1^q). \quad (42)$$

In this case, the components of the covariant metric tensor are

$$\begin{aligned} g_{qq} &= (-1+q)^{-3} \left(-2 + 2p_1^q + 2p_1^q \ln(p_1) - 2p_1^q \ln(p_1)q \right. \\ &\quad \left. + p_1^q \ln(p_1)^2 - 2p_1^q \ln(p_1)^2 q + p_1^q \ln(p_1)^2 q^2 \right), \\ g_{qp_1} &= (-1+q)^{-2} p_1^{-1+q} \left(-q \ln(p_1) + \ln(p_1)q^2 - 1 \right), \\ g_{p_1 p_1} &= p_1^{q-2} q. \end{aligned} \quad (43)$$

The determinant of the metric tensor is

$$\begin{aligned} \|g\| &= -(-1+q)^{-4} p_1^2 \left(-2p_1^q q + 2p_1^q q^2 + 2p_1^{2q} q - 2p_1^{2q} q^2 + 4p_1^{2q} \ln p_1 q \right. \\ &\quad \left. - 6p_1^{2q} \ln p_1 q^2 + 2p_1^{2q} \ln p_1 q^3 + p_1^{2q} \ln(p_1)^2 q - 2p_1^{2q} \ln(p_1)^2 q^2 \right. \\ &\quad \left. + p_1^{2q} \ln(p_1)^2 q^3 + p_1^{2q} \right). \end{aligned} \quad (44)$$

The corresponding Ricci scalar is given by

$$\begin{aligned} R &= -\frac{1}{2}(-1+q)^2 \left(-2q + 2q^2 + 2p_1^q q - 2p_1^q q^2 + 4p_1^q \ln(p_1)q - 6p_1 q \ln(p_1)q^2 \right. \\ &\quad \left. + 2p_1^q \ln(p_1)q^3 + p_1 q \ln(p_1)^2 q - 2p_1^q \ln(p_1)^2 q^2 + p_1^q \ln(p_1)^2 q^3 + p_1^q \right)^{-2} \times \\ &\quad \left(-2 - 16q^2 + 8q^3 - 22p_1^q \ln(p_1)q + 34p_1^q \ln(p_1)q^2 - 20p_1^q \ln(p_1)q^3 \right. \\ &\quad - 9p_1^q \ln(p_1)^2 q + 18p_1^q \ln(p_1)^2 q^2 - 13p_1^q \ln(p_1)^2 q^3 - 14p_1^q q + 16p_1^q q^2 \\ &\quad + p_1^q \ln(p_1)^2 + 4q \ln(p_1) + 4p_1^q \ln(p_1) + 2 \ln(p_1)^2 q^2 - 4 \ln(p_1)^2 q^3 \\ &\quad - p_1^q \ln(p_1)^3 q + 3p_1^q \ln(p_1)^3 q^2 - 3p_1^q \ln(p_1)^3 q^3 + p_1 q \ln(p_1)^3 q^4 \\ &\quad + 3p_1^q \ln(p_1)^2 q^4 - 2 \ln(p_1)q^3 + 4p_1^q - 6 \ln(p_1)q^2 - 8p_1^q q^3 \\ &\quad \left. + 2 \ln(p_1)^2 q^4 + 4 \ln(p_1)q^4 + 8q + 4p_1^q q^4 \ln(p_1) \right). \end{aligned} \quad (45)$$

For the extreme value of the entropic parameter $q = 1$, the underlying statistical configuration reduces to a non-interacting configuration, as expected, and the metric tensor diverges as depicted in Fig. (7).

2. Two particle Tsallis system

Following Eq.(41), the Tsallis entropy of a two particle system is

$$S_q^T(q, p_1, p_2) := -\frac{(1 - p_1^q - p_2^q)}{1 - q}. \quad (46)$$

The components of the thermodynamic tensor can be obtained from the above Tsallis entropy as

$$g_{qq} = \frac{1}{(1-q)^3} \left(-2 + 2p_1^q + 2p_2^q + (2p_1^q \ln(p_1) + 2p_2^q \ln(2p_2))(1-q) \right)$$

Thermodynamic metric for Tsallis one particle system

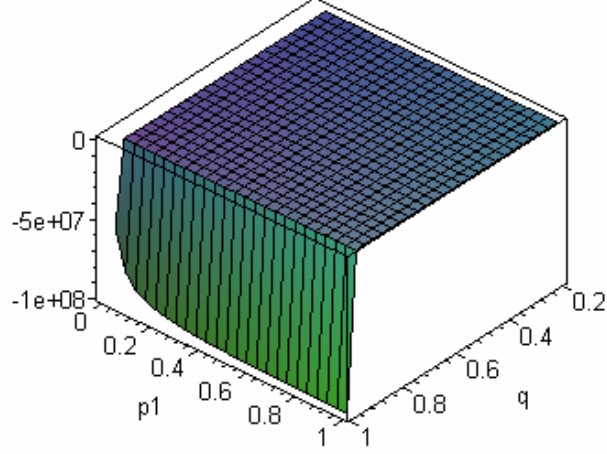


FIG. 7: The determinant of the metric tensor (DM) as a function of probability p , and the entropic parameter q , in a one particle Tsallis system.

Thermodynamic curvature for Tsallis one particle system

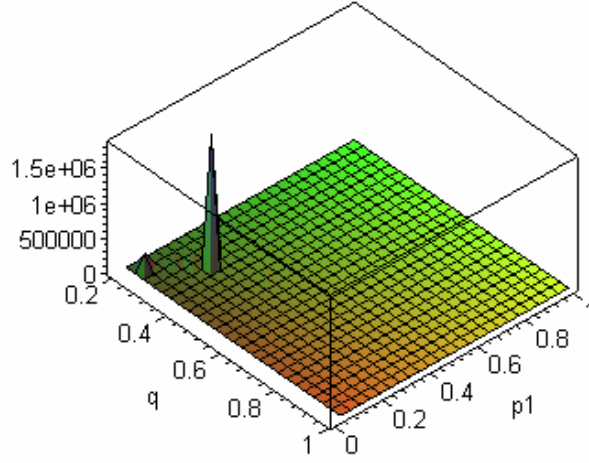


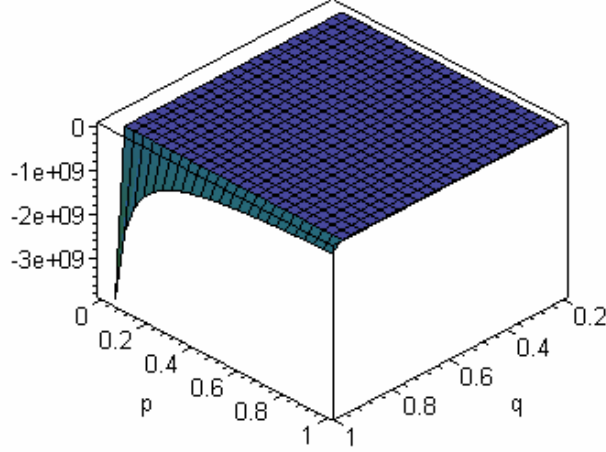
FIG. 8: Thermodynamic curvature as a function of probability p_1 , and the entropic parameter q , in a one particle Tsallis system.

$$\begin{aligned}
 & +(q^2 - 2q)(p_1^q \ln(p_1)^2 + p_2^q \ln(p_2)^2) + p_1^q \ln(p_1)^2 + p_2^q \ln(p_2)^2 \Big), \\
 g_{qp_i} &= \frac{p_i^{q-1}}{(q-1)^2} \left(-q \ln(p_i) + \ln(p_i) q^2 - 1 \right), \\
 g_{p_i p_i} &= p_i^{q-2} q.
 \end{aligned} \tag{47}$$

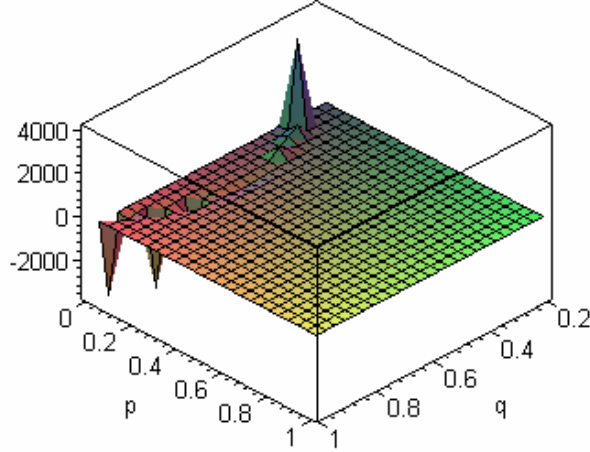
As in the case of Rényi entropy, we see in this case similar symmetries between the components of the metric tensor. The determinant of the metric tensor is

$$\begin{aligned}
 \|g\| &= q(1-q)^{-4} \Big(2p_2^{q-2} q^3 p_1^{2q-2} \ln(p_1) + p_2^{2q-2} q^3 \ln(p_2)^2 p_1^{q-2} \\
 &+ p_2^{q-2} q^3 p_1^{2q-2} \ln(p_1)^2 + 2p_2^{2q-2} q^3 \ln(p_2) p_1^{q-2} - p_2^{2q-2} q^2 p_1^{q-2} \Big)
 \end{aligned}$$

Thermodynamic metric for Tsallis two particle system

FIG. 9: The determinant of the metric tensor as a function of probability p , and the entropic parameter q , in a two particle Tsallis system.

Thermodynamic curvature for Tsallis two particle system

FIG. 10: Thermodynamic curvature as a function of probability p , and the entropic parameter q , in a two particle Tsallis system.

$$\begin{aligned}
& -6p_2^{q-2}q^2p_1^{2q-2}\ln(p_1) - 2p_2^{q-2}q^2p_1^{2q-2} + 2p_2^{q-2}q^2p_1^{q-2} \\
& -2p_2^{q-2}q^2p_1^{2q-2}\ln(p_1)^2 - 6p_2^{2q-2}q^2\ln(p_2)p_1^{q-2} - 2p_2^{2q-2}q^2p_1^{q-2}\ln(p_2)^2 \\
& + 2p_2^{q-2}qp_1^{2q-2} + p_2^{q-2}qp_1^{2q-2}\ln(p_1)^2 + 4p_2^{2q-2}q\ln(p_2)p_1^{q-2} \\
& + 4p_2^{q-2}qp_1^{2q-2}\ln(p_1) + p_2^{2q-2}q\ln(p_2)^2p_1^{q-2} + 2p_2^{2q-2}qp_1^{q-2} \\
& - 2p_2^{q-2}qp_1^{q-2} + p_2^{2q-2}p_1^{q-2} + p_1^{2q-2}p_2^{q-2} \Big). \tag{48}
\end{aligned}$$

For the case of $q = 1/3$, corresponding to the case of free particles [3], and equal probabilities $p_1 = p_2 = p$, the thermodynamic metric and curvature, in the case of two particle Tsallis system have the following forms

$$\|g\| = -\frac{27}{16} \left(\frac{40 \ln(p)}{27p^3} + \frac{8 \ln(p)^2}{27p^3} + \frac{26}{9p^3} - \frac{4}{9p^{10/3}} \right) \tag{49}$$

$$,$$
(50)

and

$$\begin{aligned}
R = & -\frac{3}{2} \left(-\frac{8}{27} \ln(p)^2 p^{1/3} + \frac{40}{27} p^{1/3} \ln(p) - \frac{4}{9} + \frac{26}{9} p^{1/3} \right)^{-2} \times \\
& \left(\frac{944}{2187} \ln(p) + \frac{112}{2187} \ln(p)^2 - \frac{64}{6561} p^{1/3} \ln(p)^4 - \frac{1216}{6561} p^{1/3} \ln(p)^3 \right. \\
& \left. - \frac{5392}{6561} \ln(p)^2 p^{1/3} - \frac{928}{729} p^{1/3} \ln(p) - \frac{616}{729} p^{1/3} + \frac{308}{729} \right).
\end{aligned}
\tag{51}$$

These are plotted in Figs. (9) and (10), respectively, for different values of q . For $q = 1$, the underlying configuration is seen to possess $R = 0$, while the determinant of the metric tensor $\|g\|$ is, again, seen to diverge. From Fig. (10) for the correlation length (thermodynamic curvature), the nature of statistical interactions show interesting features, in the form of bumps, near the limit $q = 1$.

3. Three particle Tsallis system

We now consider the three particle system whose Tsallis entropy is

$$S_q^T(q, p_1, p_2, p_3) = \frac{1}{1-q} (1 - p_1^q - p_2^q - p_3^q). \tag{52}$$

The components of the covariant metric tensor are

$$\begin{aligned}
g_{qq} = & (-1+q)^{-3} \left(-2 + 2p_1^q + 2p_2^q + 2p_3^q + 2p_1^q \ln(p_1) - 2p_1^q \ln(p_1)q + 2p_2^q \ln(p_2) \right. \\
& - 2p_2^q \ln(p_2)q + 2p_3^q \ln(p_3) - 2p_3^q \ln(p_3)q + p_1^q \ln(p_1)^2 - 2p_1^q \ln(p_1)^2 q \\
& + p_1^q \ln(p_1)^2 q^2 + p_2^q \ln(p_2)^2 - 2p_2^q \ln(p_2)^2 q + p_2^q \ln(p_2)^2 q^2 + p_3^q \ln(p_3)^2 \\
& \left. - 2p_3^q \ln(p_3)^2 q + p_3^q \ln(p_3)^2 q^2 \right), \\
g_{qp_i} = & (-1+q)^{-2} p_i^{q-1} \left(-\ln(p_i)q + \ln(p_i)q^2 - 1 \right); i = 1-3, \\
g_{p_i p_i} = & p_i^{q-2} q.
\end{aligned}
\tag{53}$$

In this case, symmetries similar to the ones seen before, are observed. The DM for the equiprobable configuration, with $p_i = p$, reduces to

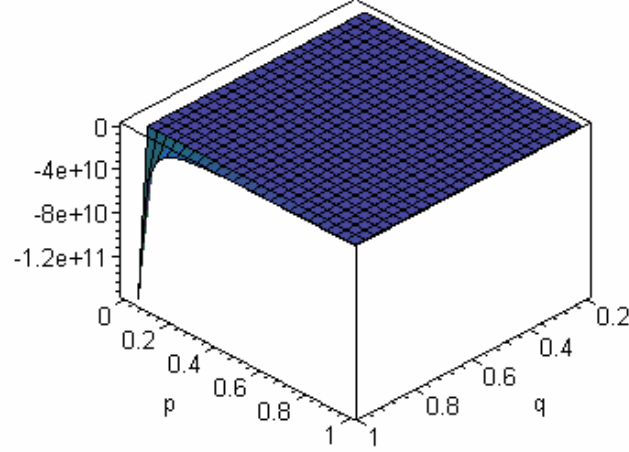
$$\begin{aligned}
\|g\| = & -q^2 p^{3q-6} (-1+q)^{-4} \left(2q^2 - 18p^q \ln(p)q^2 + 3p^q \ln(p)^2 q^3 + 6p^q \ln(p)q^3 \right. \\
& \left. + 12p^q \ln(p)q + 3p^q \ln(p)^2 q - 6p^q \ln(p)^2 q^2 + 3p^q + 6p^q q - 6p^q q^2 - 2q \right).
\end{aligned}
\tag{54}$$

This is shown in Fig. (11). In this limit, we see that the expressions for the scalar curvature, for $q = 1/3$, simplifies to

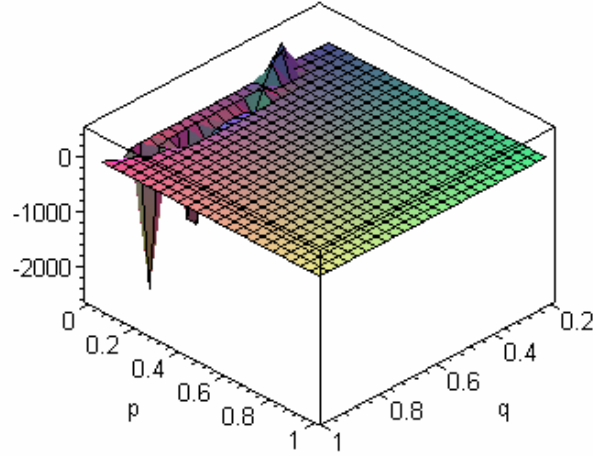
$$\begin{aligned}
R = & -\frac{3}{2} \left(-\frac{4}{9} + \frac{20}{9} p^{1/3} \ln(p) + \frac{4}{9} p^{1/3} \ln(p)^2 + \frac{13}{3} p^{1/3} \right)^{-2} \times \left(\frac{736}{729} \ln(p) \right. \\
& - \frac{760}{81} p^{1/3} \ln(p) - \frac{9520}{2187} p^{1/3} \ln(p)^2 + \frac{44}{27} - \frac{112}{2187} \ln(p)^4 p^{1/3} \\
& \left. - \frac{1888}{2187} \ln(p)^3 p^{1/3} - \frac{253}{27} p^{1/3} + \frac{80}{729} \ln(p)^2 \right).
\end{aligned}
\tag{55}$$

Finally, at $q = 1$, we find that the scalar curvature reduces to $R = \frac{1}{6p}$, while $\|g\|$ diverges. Consistent with the three particle Rényi case, here again, we find that the three particle Tsallis system has a different nature than that of the two particle one. The DM and thermodynamic curvature are shown in Figs. (11) and (12), respectively. In particular,

Thermodynamic metric for Tsallis three particle system

FIG. 11: The determinant of the metric tensor as a function of probability p , and the entropic parameter q , in a three particle Tsallis system.

Thermodynamic curvature for Tsallis three particle system

FIG. 12: Thermodynamic curvature as a function of the probability p , and the entropic parameter q , in a three particle Tsallis system.

it turns out again that there exists a degenerate metric tensor, and the underlying configuration has thermodynamic interactions, as can be inferred from the bumps in the plot of thermodynamic curvature, Fig. (12). An explanation of the origin of these interactions may, again, lie in some form of superstatistics [21, 22], where the entropic parameter q is defined by physical properties of the system, and is thus a signature of a complex system.

The DM and the scalar curvature for the one particle Tsallis configuration are shown in Figs. (7) and (8), respectively. From Fig. (7), it is clear that DM is non-zero and acquires large-negative values near $q = 1$. As expected, the curvature scalar does not show any divergence in general except at the zeros of the determinant of the metric tensor, as seen in Fig. (8), and the peaks always acquire a positive value in the chosen domain of parameters. The bumps in curvature show the presence of non-trivial interactions in the statistical configuration. Larger the height of a bump, stronger will be the interactions. In the case of a two particle configuration, the system is seen to become more stable, except for $p = 0$ and $q = 1$, as seen in Fig. (9). The curvature scalar, Fig. (10), shows both positive and negative values, which implies highly non-trivial interaction present in the system depending upon the parameter space (p, q) . Similar conclusions hold for the three particle Tsallis configurations, Figs. (11) and (12).

D. Abe Entropy

Now, we consider the Abe entropy [2, 5], which is a symmetric modification of the Tsallis entropy, an inspiration from the theory of quantum groups. It is given by

$$S_q^{Abe} = - \sum_i \frac{p_i^q - p_i^{q^{-1}}}{q - q^{-1}}. \quad (56)$$

It is related to the Tsallis entropy by

$$S_q^{Abe} = \frac{(q-1)S_q^T - (q^{-1}-1)S_{q^{-1}}^T}{q - q^{-1}}. \quad (57)$$

In the subsequent analysis, we focus our attention on systems, described by the Abe entropy.

1. Single particle Abe system

From the Eq. (56), the entropy of a single particle Abe system is given by

$$S(q, p_1) = - \frac{p_1^q - p_1^{1/q}}{q - 1/q}. \quad (58)$$

The components of the covariant metric tensor are

$$\begin{aligned} g_{qq} &= q^{-3}(q^2 - 1)^{-3} \left(p_1^q \ln(p_1)^2 q^8 - 2p_1^q \ln(p_1)^2 q^6 \right. \\ &\quad + p_1^q \ln(p_1)^2 q^4 - p_1^{1/q} \ln(p_1)^2 q^4 + 2p_1^{1/q} \ln(p_1)^2 q^2 \\ &\quad - p_1^{1/q} \ln(p_1)^2 - 4p_1^{1/q} \ln(p_1) q^5 + 4p_1^{1/q} \ln(p_1) q^3 \\ &\quad - 2p_1^q \ln(p_1) q^7 + 2p_1^q \ln(p_1) q^3 + 2p_1^q q^6 + 6p_1^q q^4 \\ &\quad \left. - 2q^6 p_1^{1/q} - 6p_1^{1/q} q^4 \right), \\ g_{qp_1} &= q^{-2}(q^2 - 1)^{-2} \left(p_1^{q-1} \ln(p_1) q^6 - p_1^{q-1} \ln(p_1) q^4 - 2p_1^{q-1} q^3 \right. \\ &\quad \left. + p_1^{-1+1/q} \ln(p_1) q^2 - p_1^{-1+1/q} \ln(p_1) + 2p_1^{-1+1/q} q^3 \right), \\ g_{p_1 p_1} &= q^{-1}(q+1)^{-1} \left(p_1^{q-2} q^3 + p_1^{-2+1/q} \right). \end{aligned} \quad (59)$$

The determinant of the metric tensor is

$$\begin{aligned} \|g\| &= -q^{-3} p_1^{-2} (q^2 - 1)^{-4} \left(4 \ln(p_1) p_1^{2/q} q^5 - 4 \ln(p_1) p_1^{2/q} q^3 \right. \\ &\quad - 8p_1^{q+1/q} \ln(p_1)^2 q^5 + 4p_1^{q+1/q} \ln(p_1)^2 q^7 - p_1^{q+1/q} \ln(p_1)^2 q^2 \\ &\quad + 6p_1^{2/q} q^4 + p_1^{2/q} \ln(p_1)^2 - p_1^{2q} \ln(p_1) q^9 + 2 \ln(p_1) q^{10} p_1^{2q} \\ &\quad + 4p_1^{q+1/q} \ln(p_1)^2 q^3 - 6p_1^{2q} q^7 + p_1^{q+1/q} \ln(p_1)^2 q^4 - 4p_1^{2q} \ln(p_1) q^8 \\ &\quad + 2q^6 p_1^{2/q} + p_1^{2q} \ln(p_1)^2 q^6 + p_1^{q+1/q} \ln(p_1)^2 q^6 + 6p_1^{q+1/q} q^3 \\ &\quad + 6p_1^{2q} q^6 + 2p_1^{2/q} q^5 + 4p_1^{2q} q^5 - 6p_1^{2/q} q^3 - 2p_1^{2q} q^9 \\ &\quad + 2p_1^{2q} q^8 + 2 \ln(p_1) q^5 p_1^{2q} - 2p_1^{2q} \ln(p_1)^2 q^8 + 2 \ln(p_1) q^6 p_1^{2q} \\ &\quad + p_1^{2/q} \ln(p_1)^2 - 2p_1^{2/q} \ln(p_1)^2 q^2 + p_1^{2q} \ln(p_1)^2 q^{10} \\ &\quad + 2q^9 p_1^{q+1/q} - 2p_1^{q+1/q} q^8 - 8p_1^{q+1/q} q^6 + 6p_1^{q+1/q} q^7 \\ &\quad \left. - 6p_1^{q+1/q} q^4 - 6p_1^{q+1/q} q^5 - p_1^{q+1/q} \ln(p_1)^2 q^8 + 4p_1^{q+1/q} q^5 \ln(p_1) \right) \end{aligned}$$

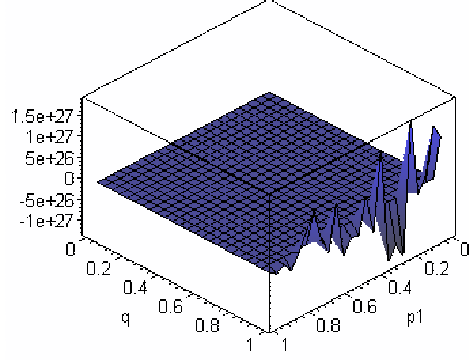


FIG. 13: The determinant of the metric tensor as a function of probability p_1 , and entropic parameter q , in a single particle Abe system

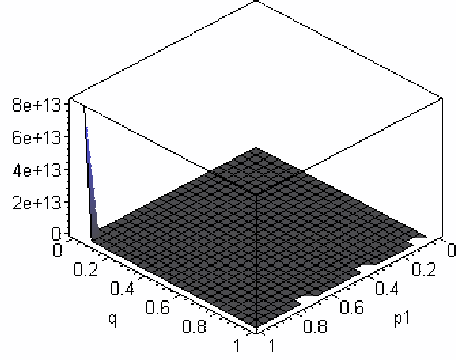


FIG. 14: Scalar curvature as a function of p_1 , and entropic parameter q , in a single particle Abe system.

$$\begin{aligned}
 &+6 \ln(p_1) q^2 p_1^{q+1/q} + 8 \ln(p_1) p_1^{q+1/q} q^8 - 2 \ln(p_1) p_1^{q+1/q} q^7 \\
 &-10 \ln(p_1) p_1^{q+1/q} q^6 - 4 \ln(p_1) q^4 p_1^{q+1/q} - 2 \ln(p_1) q^3 p_1^{q+1/q} \Big). \quad (60)
 \end{aligned}$$

It can be shown that the thermodynamic scalar curvature for $q = 1/3$ is given by

$$R = \frac{4}{9 p_1^4 g^2(p_1)} \left(\frac{4096}{177147} \right)^2 N^{(1)}(p_1), \quad (61)$$

where the numerator function $N^{(1)}(p_1)$ is

$$\begin{aligned}
 N^{(1)}(p_1) = & -\frac{39680}{129140163} p_1 - \frac{655360}{129140163} p_1^{11/3} \ln(p_1)^4 \\
 & -\frac{1863680}{43046721} p_1^{11/3} \ln(p_1)^3 - \frac{8270848}{387420489} p_1^{11/3} \ln(p_1)^2 \\
 & +\frac{18558208}{387420489} p_1^{11/3} \ln(p_1) + \frac{61952}{177147} p_1^9 \\
 & -\frac{5771264}{4782969} p_1^{19/3} \ln(p_1)^3 - \frac{27474944}{14348907} p_1^{11/3} \ln(p_1)^2 \\
 & -\frac{19625216}{14348907} p_1^{19/3} \ln(p_1) + \frac{118528}{387420489} p_1 \ln(p_1) \\
 & +\frac{5120}{43046721} p_1 \ln(p_1)^2 + \frac{4096}{387420489} p_1 \ln(p_1)^3
 \end{aligned}$$

$$\begin{aligned}
& -\frac{4096}{6561}p_1^9 \ln(p_1)^3 + \frac{10496}{19683} \ln(p_1)p_1^9 \\
& + \frac{7165952}{129140163}p_1^{11/3} - \frac{1936640}{4782969}p_1^{19/3} \\
& - \frac{1024}{2187} \ln(p_1)^2 p_1^9.
\end{aligned} \tag{62}$$

The determinant of the metric tensor is given by

$$\begin{aligned}
\|g\| = & -\frac{177147}{4096p_1^2} \left(\frac{64}{81}p_1^6 \ln(p_1)^2 + \frac{64}{59049}p_1^{2/3} \ln(p_1)^2 - \frac{32}{243}p_1^6 \ln(p_1) \right. \\
& + \frac{3584}{6561}p_1^{10/3} \ln(p_1) + \frac{128}{6561}p_1^{10/3} \ln(p_1)^2 + \frac{2264}{19683}p_1^{10/3} \\
& \left. + \frac{608}{59049}p_1^{2/3} \ln(p_1) + \frac{436}{19683}p_1^{2/3} - \frac{100}{729}p_1^6 \right).
\end{aligned} \tag{63}$$

2. Two particle Abe system

Similarly from Eq. (56), the entropy of the two particle Abe system is

$$S(q, p_1, p_2) = -\frac{p_1^q - p_1^{1/q}}{q - 1/q} - \frac{p_2^q - p_2^{1/q}}{q - 1/q}. \tag{64}$$

The components of the covariant metric tensor are

$$\begin{aligned}
g_{qq} = & q^{-3}(q^2 - 1)^{-3} \left(p_1^q \ln(p_1)^2 q^4 + 2p_2^q \ln(p_2) q^3 - p_2^{1/q} \ln(p_2)^2 q^4 \right. \\
& - 2p_1^q \ln(p_1) q^7 + 2p_1^q \ln(p_1) q^3 + 4p_1^{1/q} \ln(p_1) q^3 + 2p_1^{1/q} \ln(p_1)^2 q^2 \\
& - 2p_1^q \ln(p_1)^2 q^6 - p_1^{1/q} \ln(p_1)^2 q^4 + p_2^q \ln(p_2)^2 q^4 - 2 \ln(p_2) p_2^q q^7 \\
& - 2p_2^q \ln(p_2)^2 q^6 - 4 \ln(p_2) q^5 p_2^{1/q} + 2p_2^{1/q} \ln(p_2)^2 q^2 + 4p_2^{1/q} \ln(p_2) q^3 \\
& + p_1^q \ln(p_1)^2 q^8 - 4 \ln(p_1) p_1^{1/q} q^5 + p_2^q \ln(p_2)^2 q^8 - 2p_2^{1/q} q^6 \\
& + 2p_2^q q^6 - 6p_2^{1/q} q^4 - 2p_1^{1/q} q^6 + 6p_1^q q^4 + 2p_1^q q^6 - 6p_1^{1/q} q^4 \\
& \left. - p_2^{1/q} \ln(p_2)^2 - p_1^{1/q} \ln(p_1)^2 + 6p_2^q q^4 \right), \\
g_{qp_i} = & q^{-2}(q^2 - 1)^{-2} \left(p_i^{q-1} \ln(p_i) q^6 - p_i^{q-1} \ln(p_i) q^4 - 2p_i^{q-1} q^3 \right. \\
& \left. + p_i^{-1+1/q} \ln(p_i) q^2 - p_i^{-1+1/q} \ln(p_i) + 2p_i^{-1+1/q} q^3 \right), \quad i = 1, 2, \\
g_{p_i p_i} = & q^{-1}(q + 1)^{-1} \left(p_i^{q-2} q^3 + p_i^{-2+1/q} \right).
\end{aligned} \tag{65}$$

For the case of $q = 1/3$, and equal values of the probabilities $p_i = p$, the determinant of the metric tensor is

$$\begin{aligned}
\|g\| = & -\frac{1594323}{16384p^4} \left(\frac{872}{531441}p + \frac{640}{6561}p^{19/3} \ln(p)^2 - \frac{1216}{1594323}p \ln(p) \right. \\
& + \frac{128}{81}p^9 \ln(p)^2 + \frac{128}{1594323}p \ln(p)^2 - \frac{64}{243}p^9 \ln(p) \\
& + \frac{10816}{177147}p^{11/3} \ln(p) + \frac{640}{177147}p^{11/3} \ln(p)^2 + \frac{2368}{2187}p^{19/3} \ln(p) \\
& \left. + \frac{4328}{19683}p^{19/3} + \frac{28072}{531441}p^{11/3} - \frac{200}{729}p^9 \right),
\end{aligned} \tag{66}$$

and the associated scalar curvature is

$$R = \frac{3}{2p^8 g^2(p)} \left(\frac{1594323}{16384} \right)^2 N^{(2)}(p). \tag{67}$$

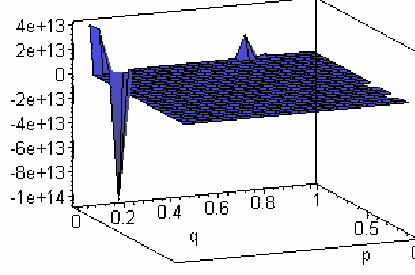


FIG. 15: The determinant of the metric tensor as a function of probability p , and entropic parameter q , in a two particle Abe system.

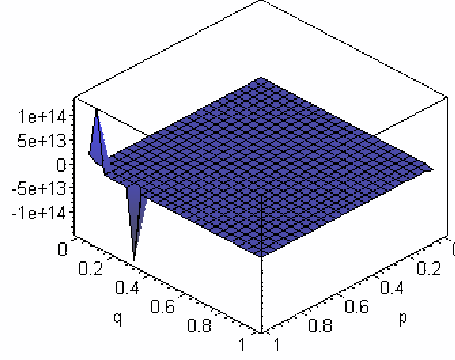


FIG. 16: Scalar curvature as a function of p and q , in a two particle Abe system.

Here, we notice that the function $N^{(2)}(p)$ may, intriguingly, be expressed as

$$N^{(2)}(p) = \sum_{a \in A} \sum_{b \in B} \alpha_{ab} \ln(p)^a p^b. \quad (68)$$

We observe further that the respective indices sets are defined as

$$\begin{aligned} A &= \{0, 1, 2, 3, 4\}, \\ B &= \left\{ 15, \frac{37}{3}, \frac{29}{3}, 7, \frac{13}{3}, \frac{5}{3} \right\}. \end{aligned} \quad (69)$$

3. Three particle Abe system

The entropy of the three particle Abe system, is given by (Eq. (56))

$$S(q, p_1, p_2, p_3, p_4) = -\frac{p_1^q - p_1^{1/q}}{q - 1/q} - \frac{p_2^q - p_2^{1/q}}{q - 1/q} - \frac{p_3^q - p_3^{1/q}}{q - 1/q}. \quad (70)$$

The component of covariant thermodynamic metric tensor in this case are

$$g_{q,q} = \left(-2 \ln(p_3) q^7 p_3^q + 4 \ln(p_3) q^3 p_3^{1/q} + 2 \ln(p_3) q^3 p_3^q + 6 p_2^q q^4 - p_1^{(1/q)} \ln(p_1)^2 \right)$$

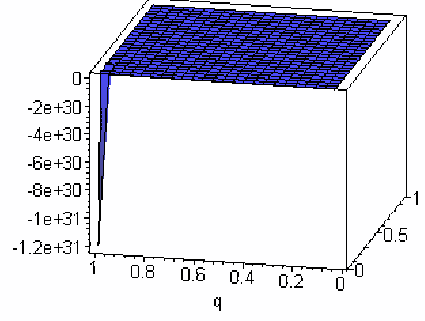


FIG. 17: The determinant of the metric tensor as a function of probability p , and entropic parameter q , in a three particle Abe system.

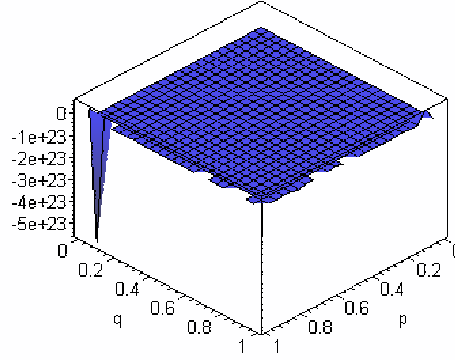


FIG. 18: Scalar curvature as a function of parameters p and q , in a three particle Abe system.

$$\begin{aligned}
& -2q^6 p_3^{1/q} - p_2^{1/q} \ln(p_2)^2 + p_1^q \ln(p_1)^2 q^8 - 2p_1^q \ln(p_1)^2 q^6 - p_1^{1/q} \ln(p_1)^2 q^4 \\
& + p_3^q \ln(p_3)^2 q^4 + p_2^q \ln(p_2)^2 q^4 + p_1^q \ln(p_1)^2 q^4 + 4 \ln(p_2) p_2^{1/q} q^3 \\
& + 2 \ln(p_2) q^3 p_2^q - 2 \ln(p_1) q^7 p_1^q + 2 p_1^{1/q} \ln(p_1)^2 q^2 - 4 \ln(p_1) p_1^{1/q} q^5 \\
& + 4 \ln(p_1) p_1^{1/q} q^3 + 2 \ln(p_1) q^3 p_1^q + p_2^q \ln(p_2)^2 q^8 - p_2^{1/q} \ln(p_2)^2 q^4 \\
& + 2 p_2^{1/q} \ln(p_2)^2 q^2 - 4 \ln(p_2) p_2^{1/q} q^5 - 2 p_2^q \ln(p_2)^2 q^6 - 2 p_3^q \ln(p_3)^2 q^6 \\
& - p_3^{1/q} \ln(p_3)^2 q^4 + 2 p_3^{1/q} \ln(p_3)^2 q^2 - 4 \ln(p_3) p_3^{1/q} q^5 + p_3^q \ln(p_3)^2 q^8 \\
& - 2 \ln(p_2) q^7 p_2^q + 2 p_3^q q^6 - 6 p_3^{1/q} q^4 - 2 q^6 p_2^{1/q} + 2 p_2^q q^6 - 6 p_2^{1/q} q^4 \\
& - 2 q^6 p_1^{1/q} + 6 p_1^q q^4 + 2 p_1^q q^6 - 6 p_1^{1/q} q^4 + 6 p_3^q q^4 - p_3^{1/q} \ln(p_3)^2 \Big) \times \\
& (q^{-3} (q^2 - 1)^{-3}), \\
g_{q, p_i} &= \left(p_i^{q-1} (\ln(p_i) q^6 - \ln(p_i) q^4 - 2 q^3) + p_i^{-1+1/q} (\ln(p_i) q^2 \right. \\
& \left. - \ln(p_i) + 2 q^3) \right) \times (q^2 (q^2 - 1)^2), \\
g_{p_i, p_i} &= \left(p_i^{q-2} q^3 + p_i^{-2+1/q} \right) \times (q (q + 1)).
\end{aligned} \tag{71}$$

The determinant of the metric tensor for $p_i \equiv p$ and $q = 1/3$ has a simple form,

$$\begin{aligned} \|g\| = & -\frac{14348907}{65536p^6} \left(p^{4/3}(a_1 + a_2 \ln(p) + a_3 \ln(p)^2) + p^4(b_1 + b_2 \ln(p) + b_3 \ln(p)^2) \right. \\ & + p^{20/3}(c_1 + c_2 \ln(p) + c_3 \ln(p)^2) + p^{28/3}(d_1 + d_2 \ln(p) + d_3 \ln(p)^2) \\ & \left. + p^{12}(e_1 + e_2 \ln(p) + e_3 \ln(p)^2) \right), \end{aligned} \quad (72)$$

where the coefficients a_i, b_i, \dots, e_i 's are

$$\begin{aligned} a_1 &= \frac{436}{4782969}, & a_2 &= \frac{608}{14348907}, & a_3 &= \frac{64}{14348907}; \\ b_1 &= \frac{25808}{4782969}, & b_2 &= \frac{7232}{1594323}, & b_3 &= \frac{512}{1594323}; \\ c_1 &= \frac{200}{2187}, & c_2 &= \frac{8960}{59049}, & c_3 &= \frac{640}{59049}; \\ d_1 &= \frac{688}{2187}, & d_2 &= \frac{3520}{2187}, & d_3 &= \frac{512}{2187}; \\ e_1 &= -\frac{100}{243}, & e_2 &= -\frac{32}{81}, & e_3 &= \frac{64}{27}. \end{aligned} \quad (73)$$

We find that the determinant of the metric tensor factorizes, and the scalar curvature may be expressed as

$$R(p) = -\frac{1}{2} \frac{N^{(3)}(p)}{g_1(p)g_2(p)g_3(p)}. \quad (74)$$

Here, the numerator function $N^{(3)}(p)$ may again be expressed as earlier for the case of a two particle Abe system (68), except that the index sets A and B are now defined as

$$\begin{aligned} A &= \{0, 1, 2, 3, 4, 5, 6\}; \\ B &= \left\{ 24, \frac{64}{3}, \frac{56}{3}, 16, \frac{40}{3}, \frac{32}{3}, \frac{16}{3}, 8, \frac{8}{3} \right\}. \end{aligned} \quad (75)$$

The factors appearing in the denominator of the scalar curvature are

$$\begin{aligned} g_1(p) &= -\frac{100}{729}p^6 + \frac{2264}{19683}p^{10/3} + \frac{64}{81}p^6 \ln(p)^2 + \frac{608}{59049}p^{2/3} \ln(p) \\ &\quad + \frac{3584}{6561}p^{10/3} \ln(p) + \frac{128}{6561}p^{10/3} \ln(p)^2 + \frac{64}{59049}p^{2/3} \ln(p)^2 \\ &\quad + \frac{436}{19683}p^{2/3} - \frac{32}{243} \ln(p)p^6, \\ g_2(p) &= \frac{64}{81}p^9 \ln(p)^2 + \frac{436}{531441}p + \frac{14036}{531441}p^{11/3} + \frac{2164}{19683}p^{19/3} \\ &\quad - \frac{100}{729}p^9 - \frac{32}{243}p^9 \ln(p) + \frac{320}{6561}p^{19/3} \ln(p)^2 + \frac{1184}{2187}p^{19/3} \ln(p) \\ &\quad + \frac{5408}{177147}p^{11/3} \ln(p) + \frac{320}{177147}p^{11/3} \ln(p)^2 + \frac{608}{1494323}p \ln(p) \\ &\quad + \frac{64}{1594323}p \ln(p)^2, \\ g_3(p) &= \frac{64}{81}p^{12} \ln(p)^2 + \frac{640}{177147}p^{20/3} \ln(p)^2 + \frac{688}{6561}p^{28/3} + \frac{7232}{4782969}p^4 \ln(p) \\ &\quad + \frac{512}{6561}p^{28/3} \ln(p)^2 + \frac{25808}{14348907}p^4 + \frac{3520}{6561}p^{28/3} \ln(p) + \frac{200}{6561}p^{20/3} \\ &\quad + \frac{512}{4782969}p^4 \ln(p)^2 + \frac{436}{14348907}p^{4/3} + \frac{608}{43046721}p^{4/3} \ln(p) \\ &\quad + \frac{64}{43046721}p^{4/3} \ln(p)^2 - \frac{32}{243}p^{12} \ln(p) + \frac{8960}{177147}p^{20/3} \ln(p) - \frac{100}{729}p^{12}. \end{aligned} \quad (76)$$

We have plotted DM, for a single particle Abe configuration in Fig. (13) and scalar curvature in Fig. (14). In this case, the system remains well defined and stable, except near $q = 1$, where there are large metric fluctuations, as seen

in Fig. (13). On the other hand, the system becomes more stable with increasing number of particles, as can be seen from the Figs. (15) and (17)), where the local correlations, as depicted by the metric tensor, can be seen to display less fluctuations (bumps in the curve) when compared to the single particle case of Fig. (13). The system is regular except near $q \sim 0.1$, as seen in Fig. (18).

E. Structural Entropy

The structural entropy has been defined as [6]

$$\begin{aligned} S_s &= S - (S_2^R) \\ &= - \sum_i p_i \ln p_i + \ln(\sum_i p_i^2), \end{aligned} \quad (77)$$

where S is the standard Shannon entropy (8) while S_2^R is the Rényi entropy (17) for $q = 2$. This entropy was studied in [23], where it was shown that an increase in the structural entropy, in the case of a tight binding model, indicates Anderson localization.

Components of the thermodynamic metric, in the case of the structure entropy for an arbitrary number of particles (say n) are as follows,

$$\begin{aligned} g_{p_i, p_j} &= \frac{4p_i p_j}{(\sum_i p_i^2)^2}, \\ g_{p_i, p_i} &= \frac{1}{p_i} - \frac{2}{\sum_i p_i^2} + \frac{4p_i^2}{(\sum_i p_i^2)^2}, \end{aligned} \quad (78)$$

where i, j will take values from 1- n . In this article we shall demonstrate the three cases, *viz.*, two particle, three particle and four particle configurations.

1. Two particle configuration

The determinant of the thermodynamic metric is

$$\|g\| = \frac{p_1^4 + p_2^4 + 2(p_1^3 + p_2^3) + 2p_1^2 p_2^2 - 2(p_2 p_1^2 + p_2^2 p_1) - 4p_1 p_2}{p_1 p_2 (p_1^2 + p_2^2)^2}, \quad (79)$$

while the scalar curvature is

$$R = -2 \left(\frac{p_1 p_2 (-2p_2 p_1^2 - 2p_2^3 + p_1^4 + 2p_1^2 p_2^2 - 2p_1 p_2^2 - 2p_1^3 + p_2^4)}{(p_1^4 + p_2^4 + 2(p_1^3 + p_2^3) + 2p_1^2 p_2^2 - 2(p_2 p_1^2 + p_2^2 p_1) - 4p_1 p_2)^2} \right). \quad (80)$$

2. Multi-particle configuration

For $p_i = p$ ($i = 1-3$), the determinant of the metric tensor and the scalar curvature are,

$$\|g\| = \frac{1}{27p^6} (27p^3 - 18p^2 - 12p + 8), \quad (81)$$

and

$$R = 2 \left(\frac{81p^4 + 216p^3 - 396p^2 + 192p - 32}{(27p^3 - 18p^2 - 12p + 8)^2} \right), \quad (82)$$

respectively. In the four particle case ($i = 1-4$), the determinant of the metric tensor is

$$\|g\| = \frac{1}{16p^8} (16p^4 - 16p^3 + 4p^2 - 1), \quad (83)$$

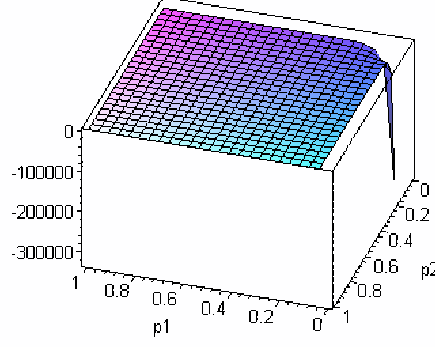


FIG. 19: Determinant of the metric tensor as a function of the probabilities p_1 and p_2 , in a two particle structural configuration.

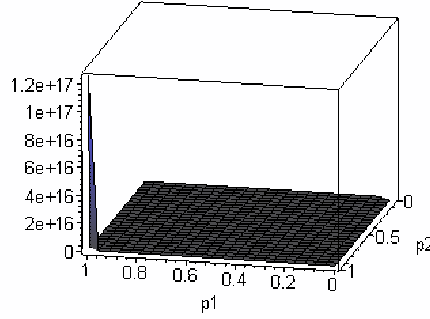


FIG. 20: Thermodynamic curvature as a function of the probabilities p_1 and p_2 , in a two particle structural configuration.

while the scalar curvature is

$$R = -\frac{1}{128} \left(\frac{384 - 3840p + 15360p^2 - 27648p^3 + 12288p^4 + 24576p^5 - 24576p^6}{(16p^4 - 16p^3 + 4p - 1)^2} \right). \quad (84)$$

The determinant of the metric tensor (DM) and scalar curvature, for a two particle structural configuration, are depicted in Figs. (19) and (20), as a function of the probabilities p_1 and p_2 . From Fig. (19), it is easy to see that the system becomes locally unstable and ill-defined near $(p_1 = 0, p_2 = 0)$. From Fig. (20), the system is seen to be globally stable and regular, except $(p_1 = 1, p_2 = 1)$. The multi-particle systems show a similar nature. The scalar curvature is well-defined and regular, except for a singularity arising from the zeros of the metric tensor, as seen in Fig. (21) for $p_i = 0.6675$.

IV. GEOMETRIC INTERPRETATION OF THE ADDITIVITY OF ENTROPIES

An interesting observation, following from the geometric analysis presented, is that the additivity of Rényi and pseudo-additivity of Tsallis entropies may be analyzed, in a simple way, by considering

$$\begin{aligned} S_{q,I,II}^R - (S_{q,I}^R + S_{q,II}^R) &= 0, \\ S_{q,I,II}^R - (S_{q,I}^T + S_{q,II}^T) &= -(q-1)S_{q,I}^T S_{q,II}^T, \end{aligned} \quad (85)$$

where, R denotes Rényi and T denotes Tsallis entropies, respectively. I and II denote the subsystems and q is the entropic parameter, as in the earlier sections. The covariant characterization, which the present analysis explores, would bring out the geometric meaning of additivity for Rényi and pseudo-additivity for Tsallis systems. More

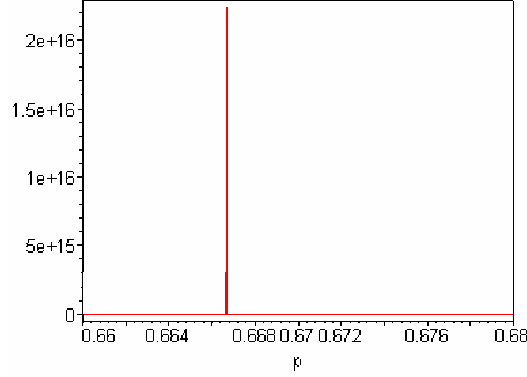


FIG. 21: Thermodynamic scalar curvature as a function of probability $p = p_i$, for all i , in a three particle structural configuration.

precisely, we can covariantly accomplish these statistical properties simply by considering the difference of local and global correlations, that is the metric tensor and thermodynamic scalar curvature, respectively.

Corresponding to the determinant of the metric tensor and the scalar curvature equations, for the Rényi and Tsallis configurations, we may realize the additivity and the pseudo-additivity at a given temperature for fixed number of probabilities. In order to have a global characterization, we define $\Delta R = R^{\text{Rényi}} - R^{\text{Tsallis}}$. It is worth mentioning that ΔR involves both the concept of additivity of the Rényi configurations and pseudo-additivity of the Tsallis configurations. Hence, for a given set of configuration parameters (q, p_i) , the plot of the quantity ΔR , as in Fig. (23), brings out the order of non-additivity, at a global level of the correlation volume of the concerned statistical system. It may further be envisaged that the quantity ΔR also indicates potential phase transitions, if any, in the chosen, finite parameter, system modulating over a range of temperatures.

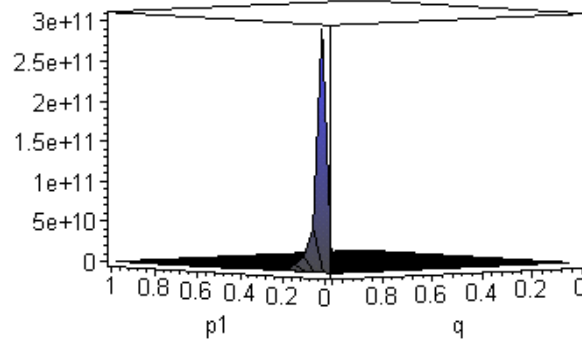
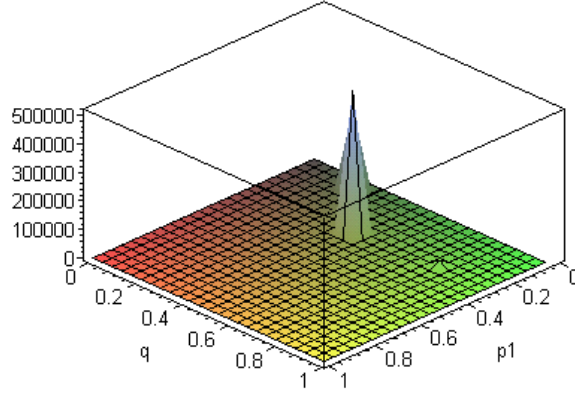
On the other hand, the difference $\Delta \|g\| = \|g\|^{\text{Rényi}} - \|g\|^{\text{Tsallis}}$, as shown in Fig. (22), would be a measure of the stability, with respect to non-additivity, of fluctuations in the configuration over the parameters under consideration. We observe that in the domain of parameters, the intrinsic geometric notion of non-additivity may locally be analyzed via the statistical correlations defined over an equilibrium distribution. In particular, the component equations $\Delta g_{ij} = g_{ij}^{\text{Rényi}} - g_{ij}^{\text{Tsallis}}$ define an order of non-additivity at an intrinsic metric level. This local information would further put forward the picture of non-additivity, as a result of local statistical fluctuations over an equilibrium thermodynamic characterization. Thus, from the present investigations, involving the calculation of the metric tensor and scalar curvatures we find that, for given entropic parameter q and probabilities p_i , the associated differences $\Delta \|g\|$ and ΔR , graphically depicted in Figs. (22), (23), respectively, bring out the nature of statistical fluctuations at the local as well as global level.

V. CONCLUSION AND FUTURE PROSPECTS

In this paper, we have applied thermodynamic geometry to open statistical systems. The metric tensor, g_{ij} , is an indicator of local correlations in the system and can be used to explore questions related to its local thermodynamic stability, while a non-zero thermodynamic scalar curvature, R , is a signature of global correlations in the system and would be useful to address questions related to phase transitions in it. Specifically, a zero scalar curvature would indicate that the components of the statistical system fluctuate independent of each other, while a divergent scalar curvature would be an indicator of phase transitions. We have analyzed the, well known, Gibbs-Shannon, Rényi and Tsallis entropies. From our motivation to understand complex statistical models, we also study the intrinsic thermodynamic geometry of Abe and Structural entropy.

Similar conclusions are valid, at a qualitative level, for the two and three parameter Rényi configurations, and is depicted in Figs. (3)-(6). The negative scalar curvature is an indicator of the attractive nature of the system, as indicated by the crusts of Fig. (4), for the two particle case. Following Fig. (6), we observe that the three parameter system has larger attraction, as it possesses larger negative valued scalar curvature.

In order to make detailed predictions, we have made an analysis for one, two and three parameter, thermally excited configurations. We find that the local statistical correlation functions associated with the Gibbs-Shannon, Rényi, Tsallis, Abe and the structural configurations are well defined, and diagonal components correspond to definite heat capacity expressions. We observe that the nature of correlations remain the same, qualitatively, as we increase

FIG. 22: $\Delta\|g\|$ as a function of p_1 and q .FIG. 23: ΔR as a function of p_1 and q

the number of parameters in either configuration.

For few parameter systems, we find that the intrinsic thermodynamic system has a non-zero scalar curvature. Physically, this shows that the generic Rényi, Tsallis, Abe and structural configurations correspond to an interacting statistical system. On the other hand, the Gibbs-Shannon entropy, having a zero scalar curvature, corresponds to a non-interacting statistical system. Thus, we are able to give a geometric meaning to the various entropies.

Further, the case of power law Hamiltonians and their quantum mechanical counterparts arise interestingly. The paper concludes with the following perspective study.

A. Power Law Hamiltonians

To extend the geometric approach, we may start with the power law Hamiltonian with its dependence on a parameter x as

$$H_i = Cx_i^\chi. \quad (86)$$

Such Hamiltonians are useful in modeling the thermostatics of complex systems [3]. When $\chi = 2$, this configuration corresponds to the one-dimensional ideal gas with quadratic hamiltonian $H = p^2/2m$. For such models [3], the Rényi

distribution is given by

$$p_i^R = Z^{-1} \left(1 - \frac{q-1}{\chi q} (C_u x_i^k - 1) \right)^{\frac{1}{q-1}}. \quad (87)$$

Here $C_u = C/U$, where C is defined as in (Eqn. 86) and U is the average energy. Substituting Eq. (87) into the Eq. (17), the Rényi entropy can be obtained for $q > q_{min} = 1/(1 + \chi)$.

For $\chi = 2$, we have extensively analyzed the thermodynamic geometric properties for these systems in Section III. In fact, the generic nature of local and global correlations, arising from the determinant of the metric tensor and scalar curvature, as depicted in the respective plots, possess a very similar nature under an addition of the extra variable p_i . It is worth mentioning, that we find similar issues to hold in various possible cases of the open statistical configurations with $\chi = 2$.

The Rényi and Shannon entropies [3] have been investigated further, for similar considerations, for $q > q_{min}$. The corresponding Tsallis, Abe and structural entropies, are expected to show similar thermodynamic geometric behavior. This analysis brings out that the generic open systems have well-defined, interacting statistical configurations.

For a given statistical system with a specific entropic parameter q , the determinants of the metric tensor and the corresponding scalar curvatures are envisaged to have definite connections with the superstatistics [3, 21, 22, 24], where the entropic parameter is defined by physical properties of a complex system, that can exchange energy and heat with a thermostat. These issues would be the subject of future investigations.

B. Quantum mechanical counterpart

As previously mentioned, the treatment followed in this paper is classical. It would be pertinent to develop the corresponding geometric treatment in the quantum regime. From this perspective [25], one may explore issues concerning entanglement, long-range global correlations and their near equilibrium behavior, from the perspective of thermodynamic geometry.

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- [1] C. Tsallis, J. Stat. Phys. **52**, 479 (1988).
 - [2] C. Beck, eprint arXiv:0902.1235.
 - [3] A. G. Bashkurov, Phys. Rev. Lett. **93**, 130601 (2004); arXiv:cond-mat/0411748.
 - [4] G. Ruppeiner, Rev. Mod. Phys. **67**, 605 (1995).
 - [5] S. Abe, Phys. Lett. A **224**, 326 (1997).
 - [6] K. Zyczkowski, Open Sys. and Information Dyn. **10**, 297 (2003).
 - [7] F. Weinhold, J. Chem. Phys. **63**, 2479 (1975), DOI:10.1063/1.431689.
 - [8] F. Weinhold, J. Chem. Phys. **63**, 2484 (1975).
 - [9] G. Ruppeiner, Phys. Rev. D **78**, 024016 (2008).
 - [10] J. Shen, Rong-Gen Cai, Bin Wang, Ru-Keng Su, Int. J. Mod. Phys. A **22**, 11-27 (2007), arXiv:gr-qc/0512035v1.
 - [11] J. E. Aman, I. Bengtsson, N. Pidokrajt, Gen. Rel. Grav. **38**, 1305-1315 (2006), arXiv:gr-qc/0601119v1.
 - [12] Jan E. Aman, I. Bengtsson, N. Pidokrajt, Gen. Rel. Grav. **35**, 1733 (2003), arXiv:gr-qc/0304015v1.
 - [13] Jan E. Aman, N. Pidokrajt, Phys. Rev. D **73**, 024017 (2006), arXiv:hep-th/0510139v3.
 - [14] T. Sarkar, G. Sengupta, B. N. Tiwari, JHEP **0611**, 015 (2006), arXiv:hep-th/0606084v1.
 - [15] B. N. Tiwari, arXiv:0801.4087v1 [hep-th], Proc. Quantum Gravity, Hoelback 2008.
 - [16] S. Bellucci, B. N. Tiwari, arXiv:0808.3921v1 [hep-th].
 - [17] T. Sarkar, G. Sengupta, B. N. Tiwari, JHEP **0810**, 076 (2008), arXiv:0806.3513v1 [hep-th].
 - [18] S. Bellucci, B. N. Tiwari, JHEP **05** (2010) 023, arXiv:0910.5314v2 [hep-th].
 - [19] S. Bellucci, V. Chandra, B. N. Tiwari, arXiv:0812.3792v1 [hep-th].
 - [20] D. Tranah and P. T. Landsberg, "Collective Phenomena", **3**, 81 (1980).
 - [21] C. Beck, Phys. Rev. Lett. **87**, 18601 (2001).
 - [22] C. Beck and E. G. D. Cohen, Physica A **322**, 267 (2003).
 - [23] J. Pipek and I. Varga, Phys. Rev. A **46**, 3148 (1992).
 - [24] A. G. Bahkurov, Physica A **340**, 153 (2004).
 - [25] J. P. Provost and G. Vallee, Commun. Math. Phys., **76**, 289 (1980).